Finding Velocity and Displacement from Acceleration

Key Ideas

- The relationships between position, velocity, and acceleration that defined velocity and acceleration in terms of derivatives can also be defined in terms of integrals.
- The change in velocity during some time interval is equal to the time integral of the acceleration over this time interval.
- The change in position during some time interval is equal to the time integral of the velocity over this time interval.
- For many real-world problems, the motion of objects is determined by integrating on a computer over time.

Learning Objectives

By the end of this section, you should:

- be able to derive the constant acceleration kinematic equations using integrals.
- use integrals to determine velocity and position from the time-dependence of the acceleration.

This section assumes you have some familiarity with integral calculus. If you have not yet been exposed to integrals, you might want to skip this section until you have become somewhat familiar with the calculus of integrals.

In the previous sections of this chapter, we introduced the kinematic properties of velocity and acceleration using derivatives. The derivative of the position function is the velocity function and the derivative of the velocity function is the acceleration function. Using integral calculus, we can work backward and calculate the velocity from the acceleration and the position from the velocity.

Kinematic Properties from Integral Calculus

Consider a particle with an acceleration $a(t)$ which is a known function of time. The time derivative of the velocity function is the acceleration

$$\frac{d}{dt}v(t) = a(t)$$

The change in the velocity of an object for some time interval, starting at a time t_i and ending at a time t_f, can be determined by integrating both sides of this equation

$$\int_{t_i}^{t_f}\left(\frac{d}{dt}v(t)\right)\,dt = \int_{t_i}^{t_f} a(t)\,dt$$

$$v(t_f) - v(t_i) = \int_{t_i}^{t_f} a(t)\,dt$$

This result comes from the fact that the integral of a derivative of a function results in the change in the function, which is a fundamental result of calculus.

Velocity $v(t)$

The velocity at any time as the integral of the acceleration plus the velocity at some initial time.

$$v(t) = \int_{t_i}^{t} a(t')\,dt' + v(t_i) \qquad \textbf{3.40}$$

The variable t' is a "dummy" integration variable to distinguish it from the final time t when we wish to determine the velocity.

Similarly, the position as a function of time can be determined from an integral of the velocity:

$$\frac{d}{dt}x(t) = v(t)$$

$$\int_{t_i}^{t_f}\frac{dt}{x}(t)\,dt = \int_{t_i}^{t_f} v(t)\,dt$$

$$x(t_f) - x(t_i) = \int_{t_i}^{t_f} v(t)\,dt$$

Displacement $x(t)$

The position at any time as the integral of the velocity plus the initial position.

$$x(t) = \int_{t_i}^{t} v(t')\, dt' + x(t_i) \qquad \textbf{3.41}$$

where t' is a dummy integration variable using the same notation as in Equation 3.40.

As an example, and a check on these results, consider the case where the acceleration is constant. Equation 3.40 and give the constant acceleration kinematic equations.

$$\begin{aligned} v(t) &= \int_{t_i}^{t} a\, dt' + v(t_i) \\ &= a\ (t - t_i) + v(t_i) \end{aligned}$$

Because the acceleration is constant, it can be pulled out of the integral, which gives the $a\,\Delta t$ term. Using the integral of the velocity to determine the position gives:

$$\begin{aligned} x(t) &= \int_{t_i}^{t} v(t')\, dt' + x(t_i) \\ &= \int_{t_i}^{t} \left(a\,(t' - t_i) + v(t_i)\right)\, dt' + x(t_i) \\ &= \frac{1}{2}\, a\,(t - t_i)^2 + v(t_i)\ (t - t_i) + x(t_i) \end{aligned}$$

In doing this integral, it's important to realize that a and $v(t_i)$ are constants. Also, in solving for the first term on the right side of the equation, a change in variable to $\tau = t' - t_i$ makes the integral very simple:

$$\int_{t_i}^{t} a\,(t' - t_i)dt' = \int_{0}^{t-t_i} a\,\tau\, d\tau = \frac{1}{2}\, a\,(t - t_i)^2$$

Example 3.18

Motion of a Motorboat

A motorboat is arriving at its dock, initially traveling at a constant velocity of $v_i = 5.0 \text{ m/s}$. To stop before it hits the dock, the motorboat accelerates in a direction opposite to its velocity until it stops. The acceleration of the motorboat is given by $a(t) = -ct$ where $c = 0.25 \text{ m/s}^3$. The units m/s^3 for the constant c are needed for the acceleration to have the correct units. (a) Determine the velocity of the motorboat as a function of time. (b) Determine the position of the motorboat as a function of time. (c) At what time from when it starts to slow down is the velocity of the motorboat equal to zero? (d) How far did the motorboat travel as it slowed to a stop?

Strategize

Because we know the acceleration as a function of time and the initial velocity, we can integrate to determine the velocity and the position.

Thinking about the integral of the acceleration, the velocity will be quadratic in time, with a t^2 term. Integrating the velocity will give a term proportional to t^3. This will help us sketch graphs of position and velocity to help define and understand the problem.

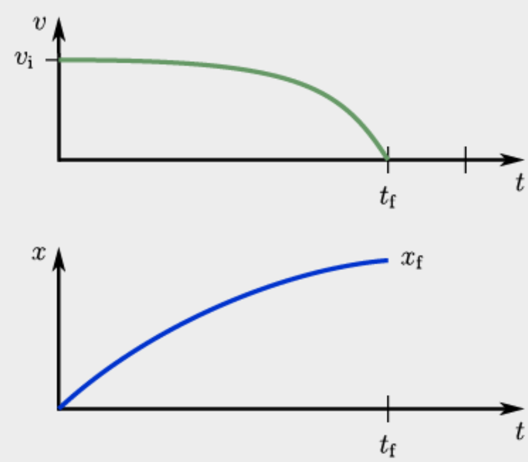

Figure 3.33 Velocity and position of a motorboat slowing to a stop with a non-constant acceleration.

Note these graphs are just sketches to see how these functions vary with time, so they do not need to be exact. The goal is to provide a picture that can help with the problem solution.

Develop and Solve

We have defined the time $t_i = 0 \text{ s}$ as the time when the motorboat begins to accelerate and $x_i = 0 \text{ m}$ as the initial position of the motorboat. Considering the known and unknown properties in this problem,

Knowns: $a(t) = -c\,t,\ v_i = 5.0 \text{ m/s},\ v_f = 0 \text{ m/s},\ x_i = 0 \text{ m}$

Unknowns: $v(t),\ x(t),\ t_f,\ x_f$